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On the Zeros of a Polynomial inside the Unit Disk

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Abstract: In this paper we find an upper bound for the number of zeros of a polynomial inside the unit disk, when the coefficients of the polynomial or their real and imaginary parts are restricted to certain conditions.
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I. INTRODUCTION AND STATEMENT OF RESULTS

Regarding the number of zeros of a polynomial inside the unit disk, the following results were recently proved by M. H. Gulzar [2] :

Theorem A: Let P(z)= $\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $\rho \geq 0$,

$$\rho + a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 .$$

Then the number of zeros of P(z) in $\frac{|a_0|}{M_1} \leq |z| \leq \delta, 0 < \delta < 1$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{2\rho + |a_n| + a_n + |a_0| - a_0}{|a_0|},$$

where $M_1 = 2\rho + |a_n| + a_n - a_0$.

Theorem B: Let P(z)= $\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients .If $\operatorname{Re} a_j = \alpha_j$, $\operatorname{Im} a_j = \beta_j$, $j = 0,1,..., n$, and for some $\rho \geq 0$,

$$\rho + \alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0 ,$$

then the number of zeros of P(z) in $\frac{|a_0|}{M_2} \leq |z| \leq \delta, 0 < \delta < 1,$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{2\rho + |\alpha_n| + \alpha_n + |\alpha_0| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|},$$

where $M_2 = 2\rho + |\alpha_n| + \alpha_n - \alpha_0 + |\beta_0| + 2\sum_{j=1}^{n} |\beta_j|$.

Theorem C: Let P(z)= $\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients .If $\operatorname{Re} a_j = \alpha_j$, $\operatorname{Im} a_j = \beta_j$, $j = 0,1,..., n$, and for some $\rho \geq 0$,

$$\rho + \beta_n \geq \beta_{n-1} \geq ... \geq \beta_1 \geq \beta_0 ,$$

then the number of zeros of P(z) in $\frac{|a_0|}{M_3} \leq |z| \leq \delta, 0 < \delta < 1,$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{2\rho+|\beta_n|+\beta_n+|\beta_0|-\beta_0+2\sum_{j=0}^{n}|\alpha_j|}{|a_0|},$$

where $M_3=2\rho+|\beta_n|+\beta_n-\beta_0+|\alpha_0|+2\sum_{j=1}^{n}|\alpha_j|$.

Theorem D. Let P (z)= $\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients such that

$$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2},\ j=0,1,..n, \text{ for some real } \beta$$

and

$$|\rho+a_n|\ge|a_{n-1}|\ge\ldots\ge|a_1|\ge|a_0|, \text{for some } \rho\ \ge 0.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{M_4}\le|z|\le\delta, 0<\delta<1,$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{(\rho+|a_n|)(\cos\alpha+\sin\alpha+1)-|a_0|(\cos\alpha-\sin\alpha-1)+2\sin\alpha\sum_{j=1}^{n-1}|a_j|}{|a_0|},$$

where

$$M_4=(\rho+|a_n|)(\cos\alpha+\sin\alpha+1)-|a_0|(\cos\alpha-\sin\alpha)+2\sin\alpha\sum_{j=1}^{n-1}|a_j|$$

In this paper, we try to give generalizations of the above results. In fact, we prove the following :

Theorem 1: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$, j=0,1,2,……,n. If for some real numbers $\lambda,\rho\ge 0$, $1\le k\le n, \alpha_{n-k}\ne 0,\ \alpha_{n-k-1}>\alpha_{n-k},\ 0<\mu\le 1,$

$$\rho+\alpha_n\ge\alpha_{n-1}\ge...\alpha_{n-k+1}\ge\lambda\alpha_{n-k}\ge\alpha_{n-k-1}\ge...\ge\alpha_1\ge\mu\alpha_0,$$

then the number of zeros of P(z) in $\frac{|a_0|}{M_5}\le|z|\le\delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{2\rho+|\alpha_n|+\alpha_n+(\lambda-1)\alpha_{n-k}+|\lambda-1||\alpha_{n-k}|-\mu(\alpha_0+|\alpha_0|+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|a_0|},$$

where $M_5=2\rho+|\alpha_n|+\alpha_n+(\lambda-1)\alpha_{n-k}+|\lambda-1||\alpha_{n-k}|-\mu(\alpha_0+|\alpha_0|)+|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|$,

and if $\alpha_{n-k}>\alpha_{n-k+1}$, then the number of zeros of P(z) in $\frac{|a_0|}{M_6}\le|z|\le\delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{2\rho+|\alpha_n|+\alpha_n+(1-\lambda)\alpha_{n-k}+|1-\lambda||\alpha_{n-k}|-\mu(\alpha_0+|\alpha_0|)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}$$

where

$M_6=2\rho+|\alpha_n|+\alpha_n+(1-\lambda)\alpha_{n-k}+|1-\lambda||\alpha_{n-k}|-\mu(\alpha_0+|\alpha_0|)+|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|$ **Remark 1:** Taking $\lambda=1,\ \mu=1$,Theorem 1 reduces to Theorem B.

Taking $\lambda=1$ in Theorem 1, we get the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$, j=0,1,2,……,n. If for some real numbers $\rho\ge 0,\ 0<\mu\le 1,$

$$\rho+\alpha_n\ge\alpha_{n-1}\ge...\ge...\ge\alpha_1\ge\mu\alpha_0,$$

then the number of zeros of P(z) in $\frac{|a_0|}{M_7} \le |z| \le \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{2\rho + |\alpha_n| + \alpha_n - \mu(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|},$$

where

$$M_7 = 2\rho + |\alpha_n| + \alpha_n - \mu(\alpha_0 + |\alpha_0|) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n} |\beta_j|$$

Remark 2: If a_j are real i.e. $\beta_j = 0$ for all j , Theorem 1 gives the following result which reduces to Theorem A by taking $\lambda = 1, \mu = 1$:

Theorem 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n .If for some real numbers $\lambda, \rho \ge 0$, $1 \le k \le n, a_{n-k} \ne 0,\ a_{n-k-1} > a_{n-k},\ 0 < \mu \le 1,$

$$\rho + a_n \ge a_{n-1} \ge ... a_{n-k+1} \ge \lambda a_{n-k} \ge a_{n-k-1} \ge ... \ge a_1 \ge \mu a_0,$$

then the number of zeros of P(z) in $\frac{|a_0|}{M_8} \le |z| \le \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{2\rho + |a_n| + a_n + (\lambda - 1)a_{n-k} + |\lambda - 1||a_{n-k}| - \mu(a_0 + |a_0|) + 2|a_0|}{|a_0|},$$

where $M_8 = 2\rho + |a_n| + a_n + (\lambda - 1)a_{n-k} + |\lambda - 1||a_{n-k}| - \mu(a_0 + |a_0|) + |a_0|$,

and if $a_{n-k} > a_{n-k+1}$, then the number of zeros of P(z) in $\frac{|a_0|}{M_9} \le |z| \le \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{2\rho + |a_n| + a_n + (1 - \lambda)a_{n-k} + |1 - \lambda||a_{n-k}| - \mu(a_0 + |a_0|) + 2|a_0|}{|a_0|},$$

where $M_9 = 2\rho + |a_n| + a_n + (1 - \lambda)a_{n-k} + |1 - \lambda||a_{n-k}| - \mu(a_0 + |a_0|) + |a_0|$.

Applying Theorem 1 to the polynomial –iP(z), we get the following result, which reduces to Theorem C by taking $\lambda = 1$:

Theorem 3: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j$ and $\mathrm{Im}(a_j) = \beta_j$, j=0,1,2,……,n. If for some real numbers $\lambda, \rho \ge 0$, $1 \le k \le n, \beta_{n-k} \ne 0,\ \beta_{n-k-1} > \beta_{n-k},\ 0 < \mu \le 1,$

$$\rho + \beta_n \ge \beta_{n-1} \ge ... \beta_{n-k+1} \ge \lambda\beta_{n-k} \ge \beta_{n-k-1} \ge ... \ge \beta_1 \ge \beta_0,$$

then the number of zeros of P(z) in $\frac{|a_0|}{M_{10}} \le |z| \le \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{2\rho + |\beta_n| + \beta_n + (\lambda - 1)\beta_{n-k} + |\lambda - 1||\beta_{n-k}| - \mu(\beta_0 + |\beta_0|) + 2|\beta_0| + 2\sum_{j=0}^{n} |\alpha_j|}{|a_0|}$$

where $M_{10} = 2\rho + |\beta_n| + \beta_n + (\lambda - 1)\beta_{n-k} + |\lambda - 1||\beta_{n-k}| - \mu(\beta_0 + |\beta_0|) + |\beta_0| + |\alpha_0| + 2\sum_{j=1}^{n} |\alpha_j|$,

and if $\beta_{n-k} > \beta_{n-k+1}$, then the number of zeros of P(z) in $\frac{|a_0|}{M_{11}} \le |z| \le \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{2\rho+|\beta_n|+\beta_n+(1-\lambda)\beta_{n-k}+|1-\lambda||\beta_{n-k}|-\mu(\beta_0+|\beta_0|)+2|\beta_0|+2\sum_{j=0}^{n}|\alpha_j|}{|a_0|},$$

where

$M_{11}=2\rho+|\beta_n|+\beta_n+(1-\lambda)\beta_{n-k}+|1-\lambda||\beta_{n-k}|-\mu(\beta_0+|\beta_0|)+|\beta_0|+|\alpha_0|+2\sum_{j=1}^{n}|\alpha_j|$.

Theorem 4: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n .If for some real numbers $\lambda>0,\rho\geq 0$, $1\leq k\leq n, a_{n-k}\neq 0,\ 0<\mu\leq 1,$

$$|\rho+a_n|\geq|a_{n-1}|\geq\ldots\geq|a_{n-k+1}|\geq\lambda|a_{n-k}|\geq|a_{n-k-1}|\geq\ldots\geq|a_1|\geq\mu|a_0|,$$

and for some real β, $|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\ j=0,1,\ldots\ldots,n$ and $|a_{n-k-1}|>|a_{n-k}|$, i.e. $\lambda>1$,then the number of zeros of P(z) in $\frac{|a_0|}{M_{12}}\leq|z|\leq\delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{\begin{array}{l}[(\rho+|a_n|(\cos\alpha+\sin\alpha+1)-|a_{n-k}|(\cos\alpha-\sin\alpha-\lambda\cos\alpha-\lambda\sin\alpha-\lambda+1)\\ -\mu|a_0|(\cos\alpha-\sin\alpha+1)+2|a_0|+2\sin\alpha\sum_{j=1,j\neq n-k}^{n-1}|a_j|]\end{array}}{|a_0|},$$

where $M_{12}=(\rho+|a_n|(\cos\alpha+\sin\alpha+1)-|a_{n-k}|(\cos\alpha-\sin\alpha-\lambda\cos\alpha-\lambda\sin\alpha-\lambda+1)$

$$-\mu|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1,j\neq n-k}^{n-1}|a_j|$$

and if $|a_{n-k}|>|a_{n-k+1}|$, i.e. $\lambda<1$, then the number of zeros of P(z) in $\frac{|a_0|}{M_{13}}\leq|z|\leq\delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{\begin{array}{l}[(\rho+|a_n|(\cos\alpha+\sin\alpha+1)+|a_{n-k}|(\cos\alpha+\sin\alpha-\lambda\cos\alpha+\lambda\sin\alpha+1-\lambda)\\ -\mu|a_0|(\cos\alpha-\sin\alpha+1)+2|a_0|+2\sin\alpha\sum_{j=1,j\neq n-k}^{n-1}|a_j|]\end{array}}{|a_0|},$$

where

$$M_{13}=(\rho+|a_n|(\cos\alpha+\sin\alpha+1)+|a_{n-k}|(\cos\alpha+\sin\alpha-\lambda\cos\alpha+\lambda\sin\alpha+1-\lambda)$$
$$-\mu|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1,j\neq n-k}^{n-1}|a_j|.$$

Remark 4: Taking $\lambda=1,\ \mu=1$, Theorem 4 reduces to Theorem D.

II. LEMMAS

For the proofs of the above results , we need the following results:

Lemma 1: . Let P(z)=$\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with complex coefficients such that $|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2}, j=0,1,..n, for$ some real β ,then for some t>0,

$$|ta_j-a_{j-1}|\leq[t|a_j|-|a_{j-1}|]\cos\alpha+[t|a_j|+|a_{j-1}|]\sin\alpha.$$

The proof of lemma 1 follows from a lemma due to Govil and Rahman [1].

Lemma 2. If p(z) is regular ,p(0) $\neq 0$ and $|p(z)| \leq M$ in $|z| \leq 1$, then the number of zeros of p(z) in $|z| \leq \delta, 0 < \delta < 1$, does not exceed $\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|p(0)|}$ (see[4],p171).

III. PROOFS OF THEOREMS

Proof of Theorem 1: Consider the polynomial

$$
\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1} z^{n-1} + \ldots\ldots + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + \ldots\ldots + (a_{n-k+1} - a_{n-k}) z^{n-k+1} + (a_{n-k} - a_{n-k-1}) z^{n-k} \\
&\quad + (a_{n-k-1} - a_{n-k-2}) z^{n-k-1} + \ldots\ldots + (a_1 - a_0) z + a_0 \\
&= -(\alpha_n + i\beta_n) z^{n+1} + (\alpha_n - \alpha_{n-1}) z^n + \ldots\ldots + (\alpha_{n-k+1} - \alpha_{n-k}) z^{n-k+1} \\
&\quad + (\alpha_{n-k} - \alpha_{n-k-1}) z^{n-k} + (\alpha_{n-k-1} - \alpha_{n-k-2}) z^{n-k-1} + \ldots\ldots + (\alpha_1 - \alpha_0) z + \alpha_0 \\
&\quad + i \sum_{j=1}^{n} (\beta_j - \beta_{j-1}) z^j + i\beta_0
\end{aligned}
$$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then

$$
\begin{aligned}
F(z) = &-(\alpha_n + i\beta_n) z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1}) z^n + \ldots\ldots + (\alpha_{n-k+1} - \alpha_{n-k}) z^{n-k+1} \\
&+ (\lambda\alpha_{n-k} - \alpha_{n-k-1}) z^{n-k} - (\lambda - 1)\alpha_{n-k} z^{n-k} \\
&+ (\alpha_{n-k-1} - \alpha_{n-k-2}) z^{n-k-1} + \ldots\ldots \\
&+ (\alpha_1 - \mu\alpha_0) z + (\mu - 1)\alpha_0 z + \alpha_0 \\
&+ i \sum_{j=1}^{n} (\beta_j - \beta_{j-1}) z^j + i\beta_0 .
\end{aligned}
$$

For $|z| \leq 1$,

$$
\begin{aligned}
|F(z)| \leq\; &|\alpha_n| + \rho + \rho + \alpha_n - \alpha_{n-1} + \ldots\ldots + \alpha_{n-k+1} - \alpha_{n-k} + \lambda\alpha_{n-k} - \alpha_{n-k-1} + |\lambda - 1||\alpha_{n-k}| \\
&+ \alpha_{n-k-1} - \alpha_{n-k-2} + \ldots\ldots + \alpha_1 - \mu\alpha_0 + (1-\mu)|\alpha_0| + |\alpha_0| + 2\sum_{j=0}^{n} |\beta_j| \\
=\; &2\rho + |\alpha_n| + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| - \mu(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|
\end{aligned}
$$

Hence by Lemma 2, the number of zeros of F(z) in $|z| \leq \delta, 0 < \delta < 1$, does not exceed

$$
\frac{1}{\log \frac{1}{\delta}} \log \frac{2\rho + |\alpha_n| + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| - \mu(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|a_0|} .
$$

On the other hand, let

$$
\begin{aligned}
Q(z) = &-a_n z^{n+1} + (a_n - a_{n-1}) z^n + \ldots\ldots + (a_{n-k+1} - a_{n-k}) z^{n-k+1} + (a_{n-k} - a_{n-k-1}) z^{n-k} \\
&+ (a_{n-k-1} - a_{n-k-2}) z^{n-k-1} + \ldots\ldots + (a_1 - a_0) z
\end{aligned}
$$

For $|z| \leq 1$,

$$
\begin{aligned}
|Q(z)| \leq\; &|\alpha_n| + \rho + \rho + \alpha_n - \alpha_{n-1} + \ldots\ldots + \alpha_{n-k+1} - \alpha_{n-k} + \lambda\alpha_{n-k} - \alpha_{n-k-1} + |\lambda - 1||\alpha_{n-k}| \\
&+ \alpha_{-k-1} - \alpha_{n-k-2} + \ldots\ldots + \alpha_1 - \mu\alpha_0 + (1-\mu)|\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n} |\beta_j|
\end{aligned}
$$

$$= 2\rho + |\alpha_n| + \alpha_n + (\lambda - 1) + |\lambda - 1||\alpha_{n-k}| - \mu(\alpha_0 + |\alpha_0|) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n}|\beta_j| = M_5 \cdot$$

Since Q(0)=0, we have, by Rouche's theorem,

$|Q(z)| \le M_5|z|$, for $|z| \le 1$.

Thus

$$|F(z)| = |a_0 + Q(z)|$$

$$\ge |a_0| - |Q(z)|$$
$$\ge |a_0| - M_5|z|$$
$$> 0$$

if $|z| < \frac{|a_0|}{M_5}$.

This shows that F(z) has no zero in $|z| < \frac{|a_0|}{M_5}$. Consequently it follows that the number of zeros of F(z) and hence P(z) in $\frac{|a_0|}{M_5} \le |z| \le \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}} \log \frac{2\rho + |\alpha_n| + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| - \mu(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

If $\alpha_{n-k} > \alpha_{n-k+1}$, then

$$F(z) = -(\alpha_n + i\beta_n)z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + ... + (\alpha_{n-k+1} - \lambda\alpha_{n-k})z^{n-k+1}$$
$$+ (\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} - (1-\lambda)\alpha_{n-k}z^{n-k+}$$
$$+ (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + ...$$
$$+ (\alpha_1 - \mu\alpha_0)z + (\mu - 1)\alpha_0 z + \alpha_0$$
$$+ i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j + i\beta_0.$$

For $|z| \le 1$,

$$|F(z)| \le |\alpha_n| + \rho + \rho + \alpha_n - \alpha_{n-1} + ...+ \alpha_{n-k+1} - \lambda\alpha_{n-k} + \alpha_{n-k} - \alpha_{n-k-1} + |1-\lambda||\alpha_{n-k}|$$
$$+ \alpha_{n-k-1} - \alpha_{n-k-2} + ... + \alpha_1 - \mu(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|$$
$$= 2\rho + |\alpha_n| + \alpha_n + (1-\lambda)\alpha_{n-k} + |1-\lambda||\alpha_{n-k}| - \mu(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|$$

Hence by Lemma 2, the number of zeros of F(z) in $|z| \le \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}} \log \frac{2\rho + |\alpha_n| + \alpha_n + (1-\lambda)\alpha_{n-k} + |1-\lambda||\alpha_{n-k}| - \mu(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

On the other hand, let

$$Q(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{n-k+1} - a_{n-k})z^{n-k+1} + (a_{n-k} - a_{n-k-1})z^{n-k}$$
$$+ (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + ... + (a_1 - a_0)z$$

$$=-(\alpha_n+i\beta_n)z^{n+1}-\rho z^n+(\rho+\alpha_n-\alpha_{n-1})z^n+......+(\alpha_{n-k+1}-\lambda\alpha_{n-k})z^{n-k+1}$$
$$+(\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}-(1-\lambda)a_{n-k}z^{n-k+1}+(\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1}+......$$
$$+(\alpha_1-\mu\alpha_0)z+(\mu-1)\alpha_0 z+i\sum_{j=1}^{n}(\beta_j-\beta_{j-1})z^j$$

For $|z|\le 1$,

$$|Q(z)|\le|\alpha_n|+\rho+\rho+\alpha_n-\alpha_{n-1}+......+\alpha_{n-k+1}-\lambda\alpha_{n-k}+\alpha_{n-k}-\alpha_{n-k-1}+|1-\lambda||\alpha_{n-k}|$$
$$+\alpha_{-k-1}-\alpha_{n-k-2}+......+\alpha_1-\mu(\alpha_0+|\alpha_0|)+|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|$$
$$=2\rho+|\alpha_n|+\alpha_n+(1-\lambda)+|1-\lambda||\alpha_{n-k}|-\mu(\alpha_0+|\alpha_0|)+|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|=M_6.$$

Since Q(0)=0, we have, by Rouche's theorem,

$|Q(z)|\le M_6|z|$, for $|z|\le 1$.

Thus

$$|F(z)|=|a_0+Q(z)|$$
$$\ge|a_0|-|Q(z)|$$
$$\ge|a_0|-M_6|z|$$
$$>0$$

if $|z|<\dfrac{|a_0|}{M_6}$.

This shows that F(z) has no zero in $|z|<\dfrac{|a_0|}{M_6}$. Consequently it follows that the number of zeros of F(z) and hence P(z) in $\dfrac{|a_0|}{M_6}\le|z|\le\delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{2\rho+|\alpha_n|+\alpha_n+(1-\lambda)\alpha_{n-k}+|1-\lambda||\alpha_{n-k}|-\mu(\alpha_0+|\alpha_0|)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

That proves Theorem 1.

Proof of Theorem 4: Consider the polynomial

$$F(z)=(1-z)P(z)$$
$$=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)$$
$$=-a_nz^{n+1}+(a_n-a_{n-1})z^n+......+(a_{n-k+1}-a_{n-k})z^{n-k+1}+(a_{n-k}-a_{n-k-1})z^{n-k}$$
$$+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+......+(a_1-a_0)z+a_0$$

If $|a_{n-k-1}|>|a_{n-k}|$, i.e. $\lambda>1$, then

$$F(z)=-a_nz^{n+1}-\rho z^n+(\rho+a_n-a_{n-1})z^n+......+(a_{n-k+1}-a_{n-k})z^{n-k+1}$$
$$+(\lambda a_{n-k}-a_{n-k-1})z^{n-k}-(\lambda-1)a_{n-k}z^{n-k}+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+......$$
$$+(a_1-\mu a_0)z+(\mu-1)a_0z+a_0$$

so that for $|z|\le 1$, we have by using Lemma 1,

$$|F(z)| \le |a_n| + \rho + |\rho + a_n - a_{n-1}| + ...+ |a_{n-k+1} - a_{n-k}| + |\lambda a_{n-k} - a_{n-k-1}|$$
$$+ |\lambda - 1||a_{n-k}| + |a_{n-k-1} - a_{n-k-2}| + ...+ |a_1 - \mu a_0| + |\mu - 1||a_0| + |a_0|$$

$$\le |a_n| + \rho + (|\rho + a_n| - |a_{n-1}|)\cos\alpha + (|\rho + a_n| + |a_{n-1}|)\sin\alpha + ...$$
$$+ (|a_{n-k+1}| - |a_{n-k}|)\cos\alpha + (|a_{n-k+1}| + |a_{n-k}|)\sin\alpha + (\lambda - 1)|a_{n-k}|$$
$$+ (\lambda|a_{n-k}| - |a_{n-k-1}|)\cos\alpha + (\lambda|a_{n-k}| + |a_{n-k-1}|)\sin\alpha$$
$$+ (|a_{n-k-1}| - |a_{n-k-2}|)\cos\alpha + (|a_{n-k-1}| + |a_{n-k-2}|)\sin\alpha + ...$$
$$+ (|a_1| - \mu|a_0|)\cos\alpha + (|a_1| + \mu|a_0|)\sin\alpha + (1 - \mu)|a_0| + |a_0|$$
$$\le (\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1)$$
$$- \mu|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|$$

Hence , by Lemma 2, the number of zeros of F(z) in $|z| \le \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}} \log \frac{[(\rho + |a_n|(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1) - \mu|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|]}{|a_0|}$$

On the other hand, let

$$Q(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{n-k+1} - a_{n-k})z^{n-k+1} + (a_{n-k} - a_{n-k-1})z^{n-k}$$
$$+ (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + ... + (a_1 - a_0)z$$

For $|z| \le 1$,

$$|Q(z)| \le (\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1)$$
$$- \mu|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|$$
$$= M_{12}.$$

Since Q(0)=0 , we have , by Rouche's Theorem,

$$|Q(z)| \le M_{11}|z|, \text{ for } |z| \le 1.$$

Thus, for $|z| \le 1$,

$$|F(z)| = |a_0 + Q(z)|$$
$$\ge |a_0| - |Q(z)|$$
$$\ge |a_0| - M_{12}|z| \qquad > 0 \quad \text{if } |z| < \frac{|a_0|}{M_{12}}.$$

This shows that F(z) has all its zeros z with $|z| \le 1$ in $|z| \ge \frac{|a_0|}{M_{12}}$.

Thus, the number of zeros of F(z) and hence P(z) in $\frac{|a_0|}{M_{12}} \le |z| \le \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{[(\rho+|a_n|(\cos\alpha+\sin\alpha+1)-|a_{n-k}|(\cos\alpha-\sin\alpha-\lambda\cos\alpha-\lambda\sin\alpha-\lambda+1) -\mu|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1,j\neq n-k}^{n-1}|a_j|]}{|a_0|}$$

If $|a_{n-k}|>|a_{n-k+1}|$, i.e. $\lambda<1$, then

$$F(z)=-a_n z^{n+1}-\rho z^n+(\rho+a_n-a_{n-1})z^n+......+(a_{n-k+1}-\lambda a_{n-k})z^{n-k+1}$$
$$+(a_{n-k}-a_{n-k-1})z^{n-k}-(1-\lambda)a_{n-k}z^{n-k+1}+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+......$$
$$+(a_1-\mu a_0)z+(\mu-1)a_0 z+a_0$$

so that for $|z|\le 1$, we have by Lemma 1,

$$|F(z)|\le|a_n|+\rho+|\rho+a_n-a_{n-1}|+......+|a_{n-k+1}-\lambda a_{n-k}|+|a_{n-k}-a_{n-k-1}|$$
$$+|1-\lambda||a_{n-k}|+|a_{n-k-1}-a_{n-k-2}|+......+|a_1-\mu a_0|+|\mu-1||a_0|+|a_0|$$

$$\le|a_n|+\rho+(|\rho+a_n|-|a_{n-1}|)\cos\alpha+(|\rho+a_n|+|a_{n-1}|)\sin\alpha+......$$
$$+(|a_{n-k+1}|-\lambda|a_{n-k}|)\cos\alpha+(|a_{n-k+1}|+\lambda|a_{n-k}|)\sin\alpha+|1-\lambda||a_{n-k}|$$
$$+(|a_{n-k}|-|a_{n-k-1}|)\cos\alpha+(|a_{n-k}|+|a_{n-k-1}|)\sin\alpha$$
$$+(|a_{n-k-1}|-|a_{n-k-2}|)\cos\alpha+(|a_{n-k-1}|+|a_{n-k-2}|)\sin\alpha+......$$
$$+(|a_1|-\mu|a_0|)\cos\alpha+(|a_1|+\mu|a_0|)\sin\alpha+(1-\mu)|a_0|+|a_0|$$
$$\le(\rho+|a_n|)(\cos\alpha+\sin\alpha+1)+|a_{n-k}|(\cos\alpha+\sin\alpha-\lambda\cos\alpha+\lambda\sin\alpha+1-\lambda)$$
$$-\mu|a_0|(\cos\alpha-\sin\alpha+1)+2|a_0|+2\sin\alpha\sum_{j=1,j\neq n-k}^{n-1}|a_j|$$

Hence , by Lemma 2, the number of zeros of F(z) in $|z|\le\delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{[(\rho+|a_n|(\cos\alpha+\sin\alpha+1)+|a_{n-k}|(\cos\alpha+\sin\alpha-\lambda\cos\alpha+\lambda\sin\alpha+1-\lambda) -\mu|a_0|(\cos\alpha-\sin\alpha+1)+2|a_0|+2\sin\alpha\sum_{j=1,j\neq n-k}^{n-1}|a_j|]}{|a_0|}$$

On the other hand, let

Q(z)= $-a_n z^{n+1}+(a_n-a_{n-1})z^n+......+(a_{n-k+1}-a_{n-k})z^{n-k+1}+(a_{n-k}-a_{n-k-1})z^{n-k}$
$+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+......+(a_1-a_0)z$

For $|z|\le 1$, by using Lemma 1,

$$|Q(z)|\le(\rho+|a_n|)(\cos\alpha+\sin\alpha+1)+|a_{n-k}|(\cos\alpha+\sin\alpha-\lambda\cos\alpha+\lambda\sin\alpha+1-\lambda)$$
$$-\mu|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1,j\neq n-k}^{n-1}|a_j|$$
$$=M_{13}.$$

Since Q(0)=0 , we have , by Rouche's Theorem,

$$|Q(z)|\le M_{13}|z|, \text{ for } |z|\le 1.$$

Thus, for $|z|\le 1$,

$$|F(z)|=|a_0+Q(z)|$$
$$\ge|a_0|-|Q(z)|$$
$$\ge|a_0|-M_{13}|z|$$

$$> 0$$

if $|z| < \dfrac{|a_0|}{M_{13}}$.

This shows that F(z) has all its zeros z with $|z| \le 1$ in $|z| \ge \dfrac{|a_0|}{M_{13}}$.

Thus, the number of zeros of F(z) and hence P(z) in $\dfrac{|a_0|}{M_{13}} \le |z| \le \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{\begin{array}{l}[(\rho + |a_n|(\cos\alpha + \sin\alpha + 1) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha + 1 - \lambda) \\ -\mu|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq n-k}^{n-1} |a_j|]\end{array}}{|a_0|}.$$

That proves Theorem 4.

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